QED Lite

Jürg Fröhlich

ETHZ-IHÉS

Simon Fest, March 2006

Results with

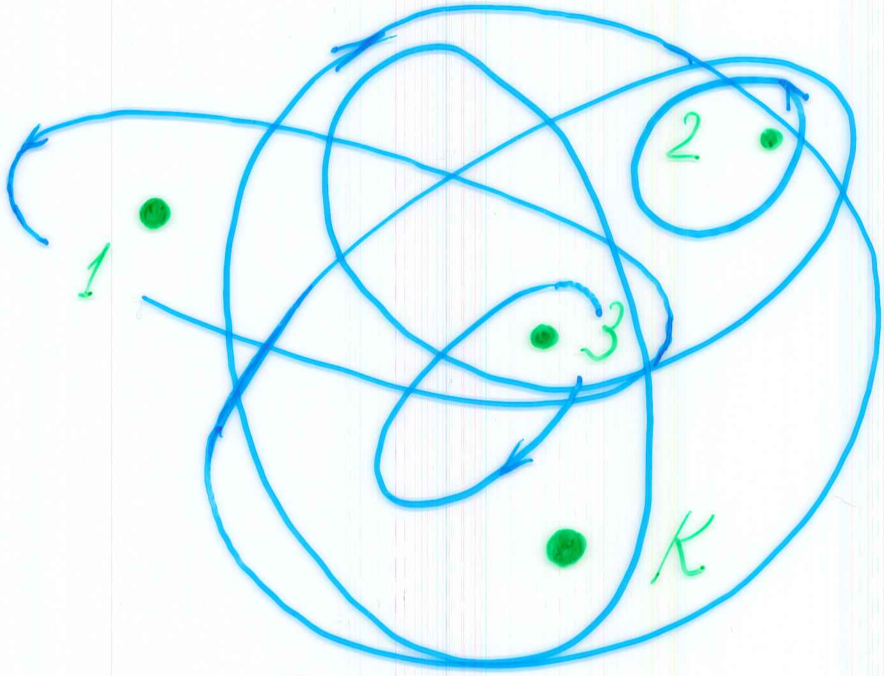
V. Bach, I.M. Sigal; L. Bugliaro, C. Fefferman, G.-M. Graf; M. Griesemer, B. Schlein; M. Merkli; A. Pizzo; T. Chen

1970–1972 / 1993–now

Siedentop; Spohn;

Matter (Heisenberg 1925)

N electrons, K nuclei
(static)



$$H_{matter} = \sum_{j=1}^{N} -\frac{\hbar^2}{2M}\Delta_j + \sum_{1\le i<j\le N} \frac{e^2}{|\vec{X}_i - \vec{X}_j|}$$

$$- \sum_{\substack{j=1 \\ \ell=1,\dots,K}}^{N} \frac{e^2 Z_\ell}{|\vec{X}_j - \vec{R}_\ell|} + \sum_{k\ne\ell} \frac{e^2 Z_k Z_\ell}{|\vec{R}_k - \vec{R}_\ell|}$$

on Hilbert space

$$\mathcal{H}_{\text{matter}} = \left(L^2(\mathbb{R}^3) \otimes \underset{\substack{\uparrow \\ \text{spin}}}{\mathbb{C}^2}\right)^{\underbrace{\wedge N}_{\text{Pauli principle}}}$$

70 years of math. research

<u>Schrödinger</u> ... Rellich, Kato, ..., Hunziker, ..., Dyson-Lenard, ..., Simon, Lieb, Thirring, ..., Enss, ... Sigal, Soffer, Graf, Derezinski, ...

em field photons

$$E = \hbar\omega\,, \quad \vec{P} = \hbar\vec{K}$$

(Planck 1900) Einstein 1905

$$\vec{E} = -\vec{\nabla}\phi + \frac{1}{c}\dot{\vec{A}}$$

$$\vec{B} = \mathrm{curl}\,\vec{A}$$

Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0$.

Fourier rep. (normal modes)

$$\vec{A}(\vec{X},t) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=\pm} \int \frac{d^3K}{\sqrt{2|\vec{K}|}} \left\{ a^*_\lambda(\vec{K}) \times \vec{\varepsilon}_\lambda(\vec{K})\, e^{i(\omega t - \vec{K}\cdot\vec{X})} + h.c. \right\}$$

$$H_f = \frac{1}{2}\int\{\vec{E}(\vec{X},t)^2 + \vec{B}(\vec{X},t)^2\}$$

$$= \sum_{\lambda=\pm}\int d^3K\, a_\lambda^*(\vec{K})(c|\vec{K}|)a_\lambda(\vec{K})$$

From now on, set $c = 1$, $\hbar = 1$, $\frac{e^2}{\hbar c} =: \alpha \simeq \frac{1}{137}$;

introduce dimensionless variables:

$\vec{X} = \beta\vec{x}$, $\vec{R} = \beta\vec{r}$, $\vec{K} = \mu\vec{k}$,

$[\beta] = [\mu^{-1}] = \text{length}$;

$\beta = (2m\alpha)^{-1}$, $\mu = 2m\alpha^2$, $\beta\mu = \alpha$,

m: phys. electron mass.

v

Quantum theory of em field: (Einstein, Jordan, Dirac, ...)

Normal modes of em field, labelled by pol. λ, wave vector $\vec{k}$, are <u>harmonic oscillators</u> with frequ. $\omega = |\vec{k}|$ ($c = 1$!). $\longrightarrow$

Quantize canonically (Heisenberg 1925):

$$[a^{\#}_{\lambda}(\vec{k}), a^{\#}_{\lambda'}(\vec{k}')] = 0,$$

$$[a_\lambda(\vec{k}), a^*_{\lambda'}(\vec{k}')] = \delta_{\lambda\lambda'}\delta(\vec{k}-\vec{k}')$$

Vacuum : Ω , $\langle \Omega, \Omega \rangle = 1$,

$$a_\lambda(\vec{k})\Omega = 0, \ \forall \lambda, \vec{k}.$$

$$\overline{\langle \sum \prod a^*(h_j)\Omega \rangle} =: \mathcal{F}_\gamma$$

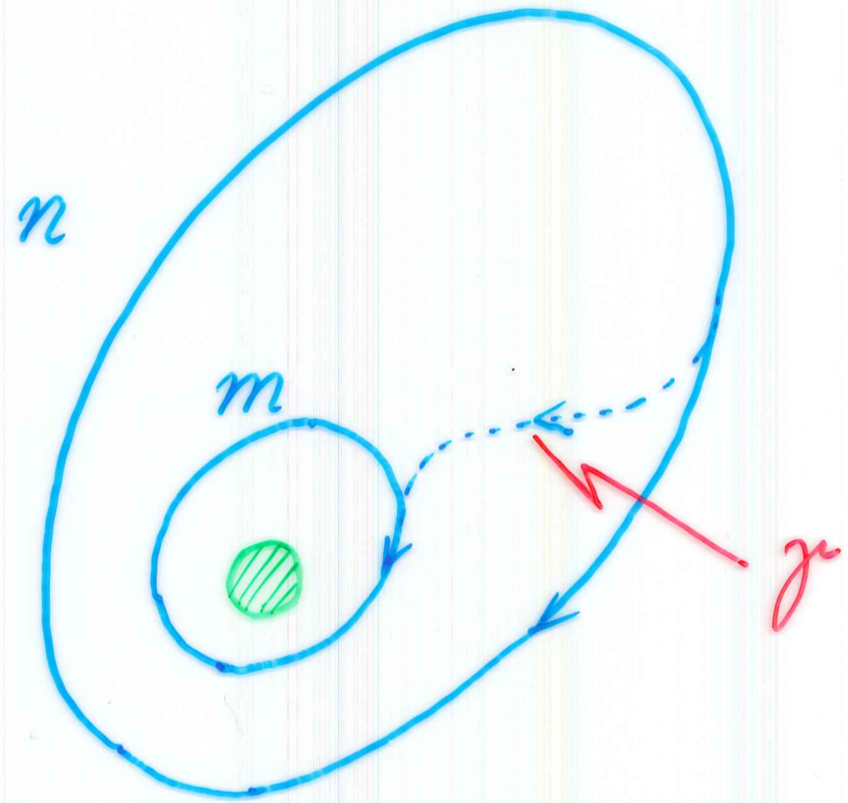
<u>Fock space</u> of photons.

$\downarrow$

Theory of quantized, free em field.

Matter interacting with (quantized) em field?

Bohr (1913), Sommerfeld, ... Einstein (1917), ..., Heisenberg (1925), ...



$$\hbar\omega_{nm} = E_n - E_m \qquad (1)$$

What are these "quantum jumps"; what is status of Bohr's eq. (1), ... ?

Prior to ~1993: No math. rigorous results on atomic spectroscopy, Rayleigh scattering, Compton effect, ….

Knew a lot about "dark" atoms, but very little about "shiny" atoms.

I will report on progress made since 1993.

(1) STANDARD MODEL OF ATOMS INTERACTING WITH EM FIELD

Hilbert space of total system is

$$\mathcal{H} := \left(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2\right)^{\wedge N} \otimes \mathcal{F}_\gamma$$

Dynamics given in terms of Hamiltonian $H_{\alpha,\Lambda}$ coupling electrons to em field;

α : strength of coupling,

Λ : UV cutoff.

$$g_{bare} = 2\,!$$

$$H_{\alpha,\Lambda} = \sum_{j=1}^{N} \left\{ \frac{1}{M_\Lambda} \left[\vec{\sigma}_j \cdot \left(-i\vec{\nabla}_j + \alpha^{3/2} \vec{A}_\Lambda (\alpha \vec{x}_j) \right) \right]^2 - \mu_\Lambda \right\}$$

$$+ V_{Coulomb}(\underline{x}, \underline{r}) + H_f \, ,$$

M_Λ : "bare" mass of e^-,

μ_Λ : chem. potential,

$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$: Pauli matrices,

$$\vec{A}_\Lambda(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=\pm} \int \frac{d^3k}{\sqrt{2|\vec{k}|}} \chi_\Lambda(\vec{k}) \times$$

$$\times \left\{ a_\lambda^*(\vec{k}) \vec{\varepsilon}_\lambda(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} + h.c. \right\},$$

$$V_{Coulomb} = \sum_{i<j} \frac{1}{|\vec{x}_i - \vec{x}_j|} - \sum_{i,\ell} \frac{Z_\ell}{|\vec{x}_i - \vec{r}_\ell|} + \cdots \, ,$$

$$H_f = \sum_{\lambda=\pm} \int d^3k \, a_\lambda^*(\vec{k}) |\vec{k}| a_\lambda(\vec{k})$$

$$\varkappa_\Lambda(\vec{k}) \overset{e.g.}{=} (2\pi\Lambda^2)^{-3/2} \exp\left(-\frac{|\vec{k}|^2}{2\Lambda^2}\right)$$

Problems. $H_{\alpha,\Lambda} = H^*_{\alpha,\Lambda}$, spec$(H_{\alpha,\Lambda})$, props. of $e^{-itH_{\alpha,\Lambda}}$, scattering th., $\lim \Lambda \to \infty$, ... ?

i) $\Lambda \to \infty$

- no positrons → no vacuum polarization ⇒ $\alpha_\Lambda \equiv \alpha$ independent of Λ.
- $\mu_\Lambda \sim M_\Lambda^{-1} \Lambda^{2-\delta}$, $0 \leq \delta \sim$?
- $M_\Lambda \sim \Lambda^{-8\pi\alpha + O(\alpha^2)}$ (perturbative RG)
- Lamb shift finite, as $\Lambda \to \infty$.

. . . .

No nonperturbative results on $\Lambda \to \infty$, yet!

From now on, fix $\Lambda \sim M_\Lambda \sim m < \infty$.

ii) <u>Stability for fixed Λ.</u>

(Lieb et al., BFG, FFG, ...)

K nuclei, $Z_\ell \leq Z$, $\forall \ell$

N electrons

Groundstate energy $\underset{K\to\infty}{\geq} -C_\Lambda K$

(class. field: $\Lambda \to \infty$ understood)

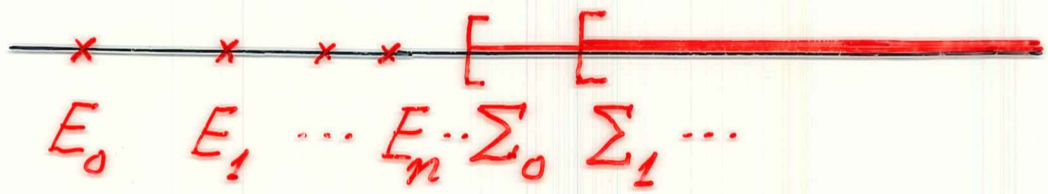
$$g_{eff} \equiv g := (\alpha \Lambda m^{-1})^{3/2} \ll 1.$$

$$H_{\alpha,\Lambda} \equiv H_g = \underbrace{H_{atom} + H_f}_{H_0} + W_g \quad (2)$$

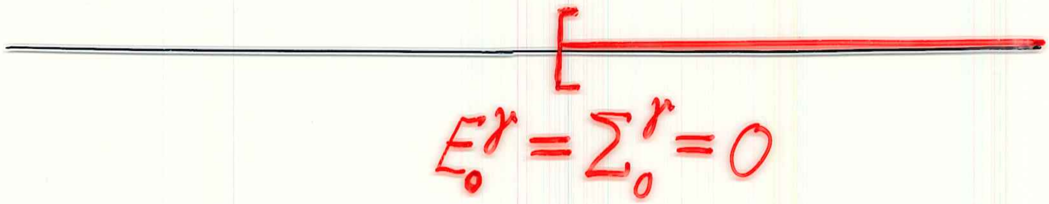
$$\mathrm{spec}(H_g) \sim ?$$

Consider e.g. single atom + radiation field.

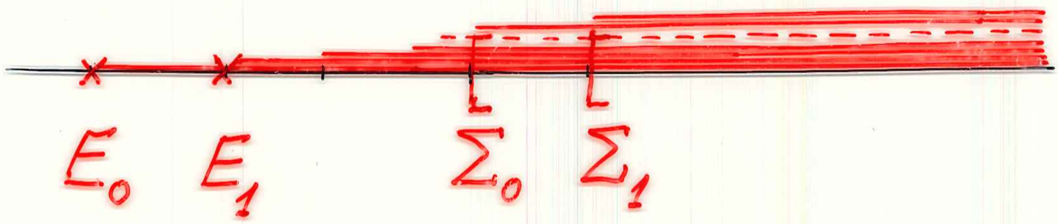
$\mathrm{spec}(H_{atom})$



$\mathrm{spec}(H_f)$



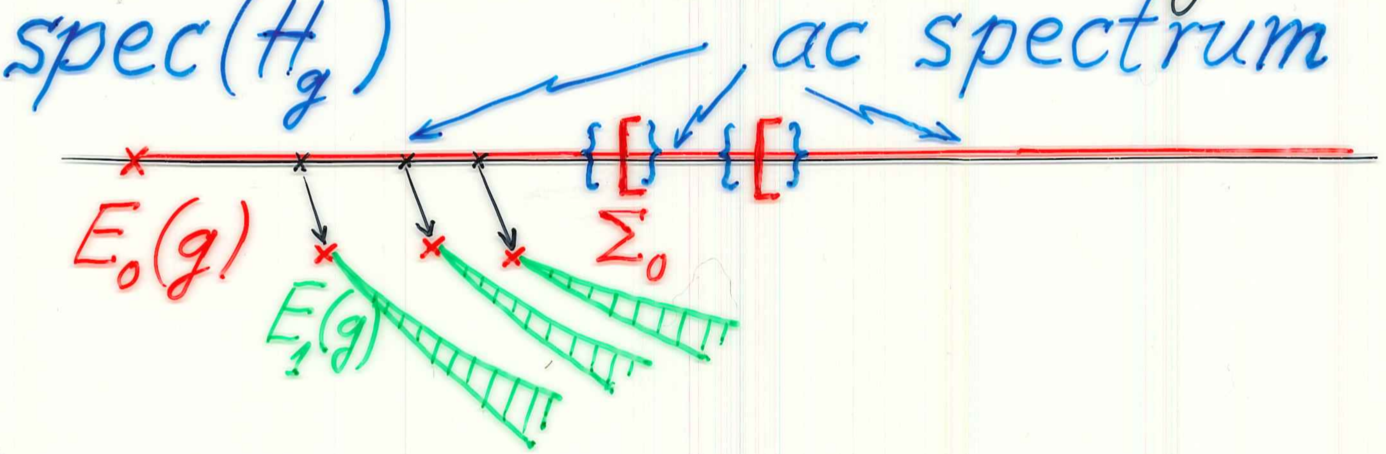
$\mathrm{spec}(H_0)$



$\mathrm{spec}(H_0)$ contains <u>embedded e.v.'s</u> $(E_1, E_2, \cdots)$ at <u>thresholds of cont. spect.</u> (from H_f).

Standard pert. th. inapplicable!

Results when pert. W_g is turned on; (Bach-F-Sigal)



$E_i(g)$: resonance energies at tip of "branch cut".

(1) stable groundstate:

$$\sigma_{pp}(H_g) = \{E_0(g)\}, \text{ below } \Sigma_0$$

(B-F-S, G-L-L, F ← G-J / B-F-P)

"bound states": For $I \subset (-\infty, \Sigma_0)$, g small enough, $\exists\, \xi < \infty$ such that

$$\| e^{|\vec{x}|/\xi} \otimes \mathbb{1}_{\mathcal{F}} \, \chi_I(H_g) \| < \text{const}_I$$

expo. decay in $\vec{x}$!

(2) $\mathrm{spec}(H_g) \setminus \{E_0(g)\}$ is purely ac, except (?) near Σ's.

(3) $\mathrm{Re}\, E_i(g) \sim$ Bethe

$\mathrm{Im}\, E_i(g) \sim$ Fermi $\quad i = 1, 2, \dots$

+ <u>cuts</u>

(4) <u>Scattering theory</u>

B - Heisenberg - Wheeler ...

(D - G, F - Griesemer - Schlein, [Spohn])

(i) <u>Ex. of asymp. e.m. fields:</u>

$$e^{iH_g t} e^{-iH_f t} a^{\#}(f) e^{iH_f t} e^{-iH_g t} \xrightarrow[t \to \pm\infty]{s} a_{\pm}^{\#}(f)$$

on bound states (or ones where <u>prop. vel. of e^-'s $< c$</u>)

(ii) AC for Rayleigh scattering

Let ψ be any bound state; $\psi_0(g)$ groundstate $(g \neq 0)$. Then, (with arb. tiny IR cutoff)

$$\psi = s\text{-}\lim \sum_{\alpha} a^*_{\pm\,(-)}(f_1^{\alpha}) \cdots a^*_{\pm\,(-)}(f_{n_\alpha}^{\alpha})\,\psi_0(g)$$

$\langle N_\gamma \rangle_{\psi_t}$?

(iii) Cor.: Relaxation to g.s.

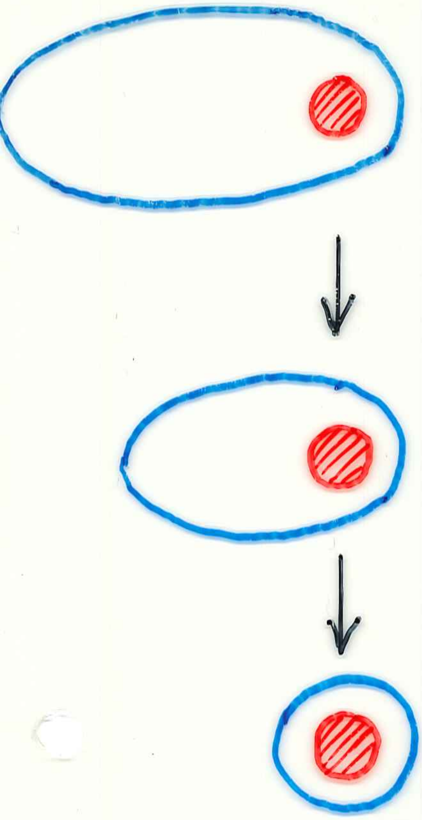
If ψ is a bound state, and A any "local observable" then

$$\langle e^{-itH_g}\psi, A e^{-itH_g}\psi \rangle \xrightarrow[t \to \pm\infty]{} \langle \psi_0(g), A\,\psi_0(g) \rangle$$

Isolated atom always relaxes to groundstate!

$$|\langle \psi_\alpha, e^{-itH_g}\psi_\alpha\rangle|$$

$$\lesssim a e^{-tg^2\Gamma_\alpha} + O_N(g^4)\,t^{-N}$$



ψ_α

$\longrightarrow \gamma$ $\quad h\nu_{\alpha\beta} \approx E_\alpha - E_\beta$

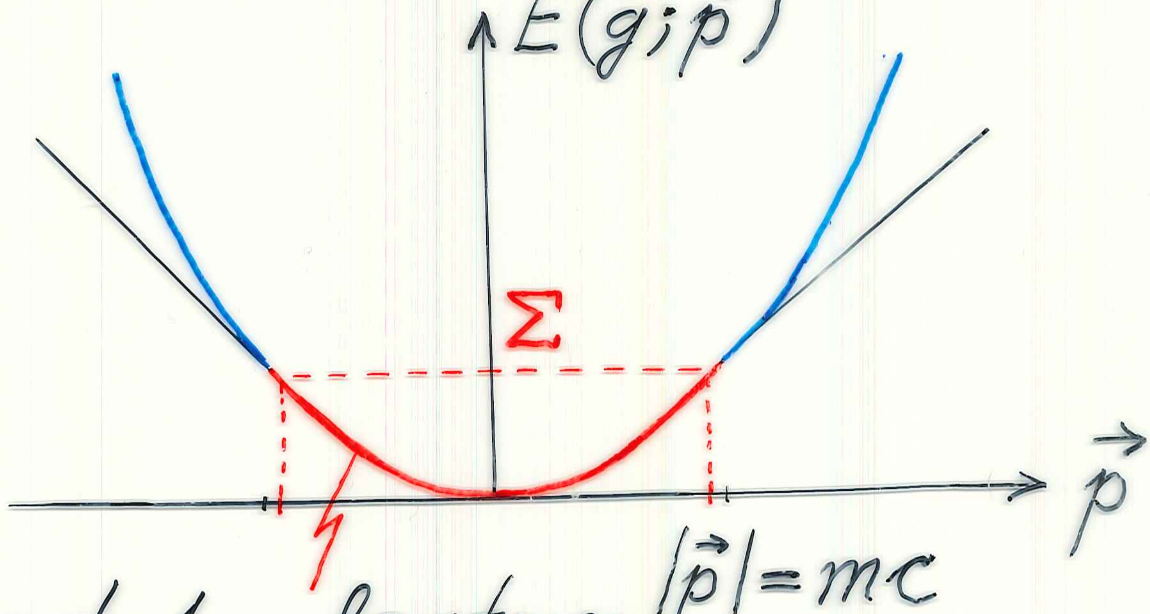
ψ_β $\quad$ (Ri-Ryd, Bohr)

$\longrightarrow \gamma$ $\quad$ from S-matrix

ψ_0 $\quad$ No (quasi-)per. int

(Bach-K-Z, J.F., A.P., ...)

(iv) AC for Compton scattering



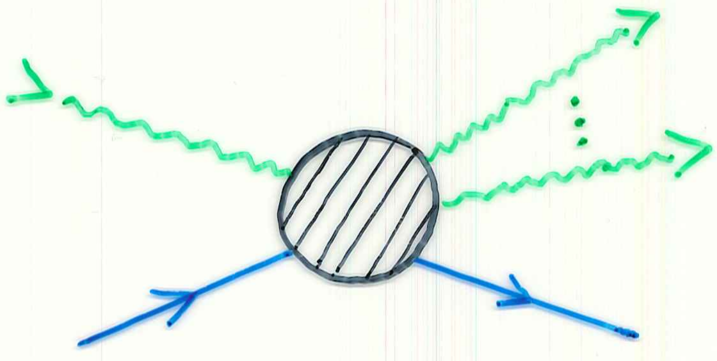
dressed 1-electron state $\psi_{DES}(\vec p)$, energy

$$E(g,\vec p) = \frac{1}{2m_{REN}}\vec p^{\,2} + O(\vec p^{\,4})$$

Consider one (freely moving) e^-; no nuclei; arb. tiny IR cutoff. Let ψ be arb. state with tot. energy below Σ; g small enough (dep. on Σ). Then

$$\psi = s\text{-}\lim \sum_\alpha \prod_1^{n_\alpha} a^*_{\pm(-)}(f_i^\alpha)\, \psi_{DES}(h),$$

$(\mathrm{suppt}\ h \subset \{\vec{p}\ |\ |\vec{p}| < \gamma mc\, !)$



Genuine IR problem when IR cutoff $\to 0$! (F, Pizzo)

BCFS : Exp. for $m_{ren.}$

Removal of IR cutoff

$$\sigma \to 0$$

(1) Ex. of dressed one-e^- states : ✓ (J.F., 1972, ...)

IR rep. is coherent state rep. (... T.C., A.P.)

Regularity of $E_g(p)$ (up to 2 derivatives)

(2) Møller wave ops. for Compton scattering & "Bremsstrahlung" : ✓ (J.F., ..., A.P.)

(3) Infrared-finite, "finite" algorithms to calculate $E_0(g), \cdots$, based on multi-scale analysis:

$$E_0(g) = E_0 + \sum_{j=1}^{N} \varepsilon_j(g)\, g^j + O(g^{N+\varepsilon})$$

$\varepsilon_j(g) = o(g^{-1})$ compt. in terms of finite # of conv. integrals; but infrared logs in g! (B-F-P)

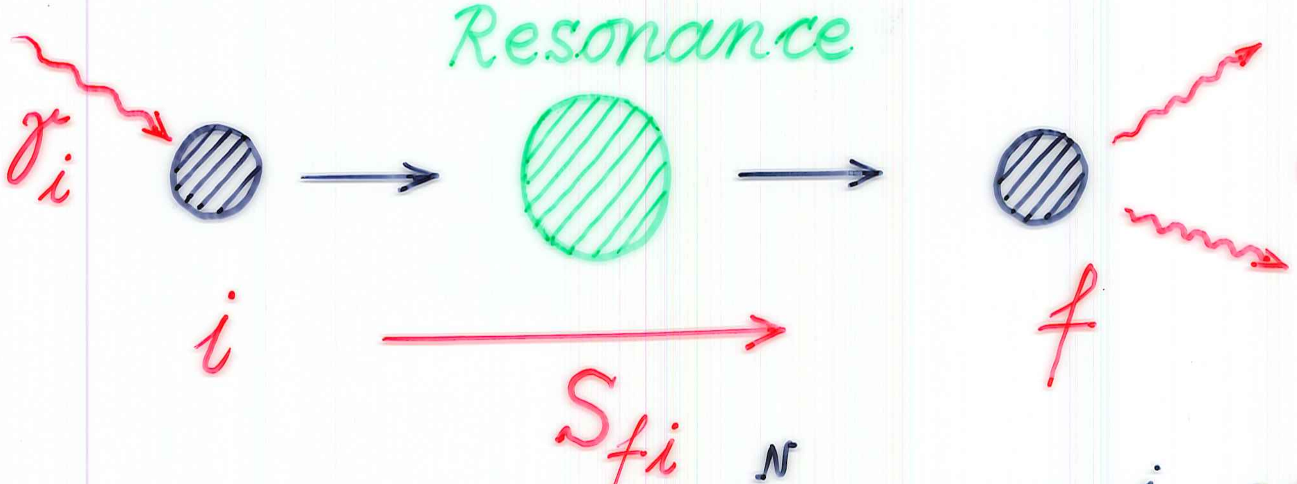
$E_0(g)$ pres. not C^∞ in g, $|g| < g_*$!

(4) Infrared-finite algor. for scattering amplitudes

of Rayleigh – (& Compton – ?) scattering :

i : atom in groundst., $\gamma^{\swarrow}$ in h_i

f : " " " , $\gamma^{\nearrow}$ in h_f



$$S_{fi}(g) = \langle f | i \rangle + \sum_{j=1}^{N} \underset{\uparrow \atop \text{IR-logs}}{\sigma_{fi}(g)}\, g^j + O(g^{N+\varepsilon})$$

→ Derivation of (Einstein–) Bohr frequency cond., … !

But no info. on Lamb shift & life times, yet, because spread in energy of states h_i and h_f of γ_i, γ_f, resp., is $O(1) \rightarrow$ not suff. accurate for Lamb shift, ...

Fundamental problems:

- Control of $e^{-i(tH_g)/\hbar}$ up to times $t \gtrsim O(g^{-2})$, $\nearrow$ E, S, Y.
- Show that, for Ψ in range of $\chi(H_g < \Sigma_0)$, $\Psi \in D(N_\gamma^{1/2})$,

$$|\langle e^{-i(tH_g)/\hbar}\Psi, N_\gamma e^{-i(tH_g)/\hbar}\Psi\rangle| \leqslant \text{const.}, \ \forall t.$$

2. RETURN TO EQUILIBRIUM & THERMAL IONIZATION

(1) R to E Consider finitely many "toy atoms" with finite-dim. state space coupled to black-body radiation at temp. $T > 0$; (Einstein 1917). Suppose "dipole matrix elements" $x_{\alpha\beta} \neq 0$, for some $\beta < \alpha$, $\forall \alpha > 0$; (where α, β label atomic eigenstates).

Let ρ be an arb. initial state of coupled system which is "normal" w.r. to <u>equilibrium state</u> ρ_f^β of black-body radiation, (i.e., very far from atoms, ρ is $\simeq$ thermal state w. temp. $T = (k_B \beta)^{-1}$).

Then, for suff. small $|g|$, $\forall T > 0$,

$$\text{"}\lim_{t \to \infty} \rho\left(e^{itH_g} A e^{-itH_g}\right) = \rho^\beta(A)\text{"},$$

where A is arb. "local obs.", and ρ^β is unique equ. state of coupled system.

R to E, as $t \to \infty$, is expo fast, for some "realistic" couplings;

$$\rho^\beta\big|_{atom} = Z_\beta^{-1} \sum_\alpha e^{-\beta E_\alpha} P_\alpha + O(|g|)$$

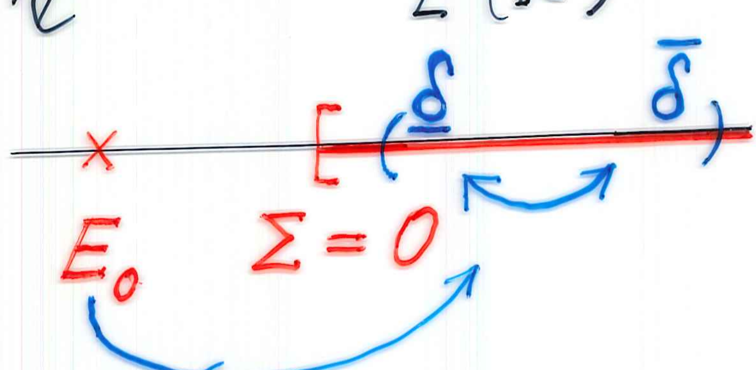
---

(2) Thermal ionization

Couple more "realistic" atom, with coexistence of point- and cont. spectrum, to black-body radiation at $T>0$.

$$H_{atom} \simeq E_0\big|_{\mathbb{C}} \oplus (-\Delta)\big|_{L^2(\mathbb{R}^3)}$$

$\mathrm{spec}(H_{atom})$



Interaction W_g assumed

to only make transitions between bound state ψ_0 (E_0) and states ψ_E with energy $E \in [\underline{\delta}, \overline{\delta}]$, $0 < \underline{\delta} < \overline{\delta} < \infty$.

Then, for $|g|$ small, $T > 0$,

$$\lim_{t \to \infty} \rho\left(e^{itH_g} P_0 e^{-itH_g}\right) = 0,$$

for an arb. initial state ρ normal w.r.to ρ_f^β;

$\nexists$ any stationary state!

Atom always entirely ionized!

OPEN PROBLEMS

Fate of electrons, after ionization?

Conjecture: "Quantum Brownian motion"

$$\rho\left([\vec{x}(t)-\vec{x}(0)]^2\right) \underset{t\to\pm\infty}{\sim} D_{\beta,g}\,|t|,$$

for some $0 < D_{\beta,g} < \infty$.

($\nearrow$ L. Erdös)

Other results on ionisation:

- by laser pulses (F-K-S; F-Schlein: Kramer's model)

Need non-pert. methods to study large-field QED, low-T, long-t behaviour

3. CONCLUSIONS

(1) Discovery of QM of atoms, etc. by (B, E) Heisenberg, B, J, Dirac, Schrödinger is miracle: Approx. valid data & rules, phys. "wrong" idealizations, ... led to correct, consistent theory.

(2) Quantum theory of atoms, ... coupled to radiation field (spectroscopy, qu. optics, lasers) still rich in math. (analytical) challenges! IR probl. large $\vec{E}, \vec{B}$

(3) Results described in this lecture lead to understanding of:

dissipation & friction through disp. radiation → irreversible behaviour

Relaxation to g.s., R to E, TI, approach to NESS, Decay of resonances (metastable states), ...

Why interesting (or important)? Foundations of thermodynamics

(4) Irreversible behaviour from reversible dynamics.
Retrieval of info. about nature through experiments always acc. by dissipation & friction, irrev. beh. ($S\uparrow$), decay of metastable states.
Results may help to develop a "quantum th. of exp.", with hope that QM (of open systs. w. ∞ many degs. of freedom) will provide its own interpretation!

---

4. SOME IDEAS OF PROOFS

(1) Ex. of stable groundstates

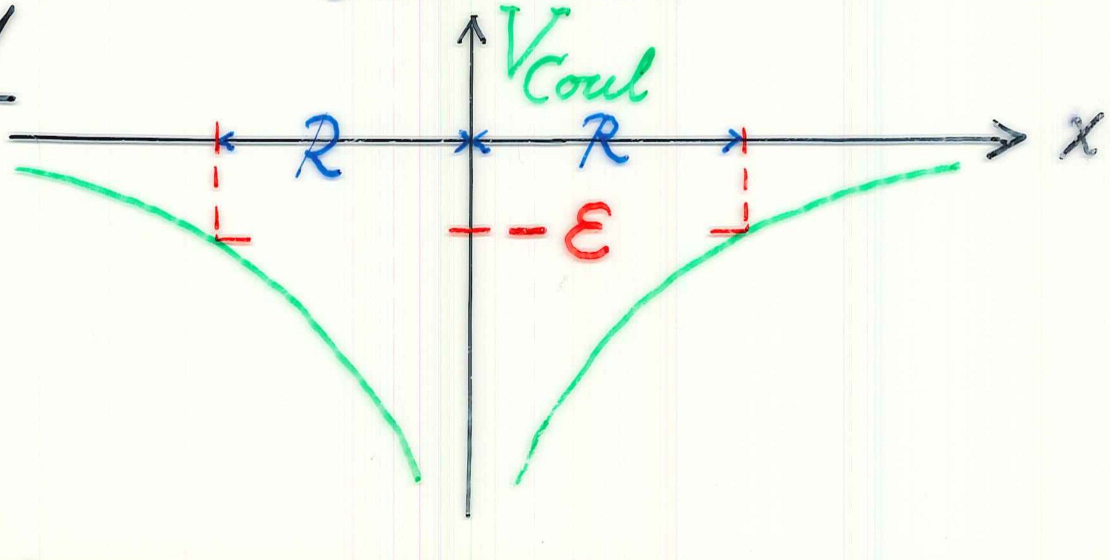
(unique, up to ↑↓, for $N=1$)

No "IR catastrophe" for boundstates, $\psi \leftrightarrow$ "soft-γ bounds": $\langle N^\gamma \rangle_\psi \leq C_\psi g^2 \int \frac{d^3k}{|\vec{k}|}$

→ "compactness arg.", (F '72 ← G-J).

Expo. decay of boundstates

$N=1$



ψ bd. state, max. energy $-E_\psi < 0$

R so large that $V_{Coul}(|\vec{x}| \geq R) \geq -\varepsilon \gg -E_\psi$

Then

$$-E_\psi \geq \langle \psi, H_g \psi \rangle = \langle \chi_< \psi, H_g \chi_< \psi \rangle$$

$$+ \langle \chi_> \psi, H_g \chi_> \psi \rangle - \varepsilon$$

$$\geq -|E_0(g)| \, \|\chi_< \psi\|^2 - 2\varepsilon - a|g|$$

$$\Rightarrow \|\chi_< \psi\|^2 \geq \frac{E_\psi - 2\varepsilon - a|g|}{|E_0(g)|} \cdots \geq 1 - o(1)$$

[$\Rightarrow$ Expo. decay of ψ from standard resolvent estimates!]

(2), (3) Nature of spec(H_g); resonances

Dilatation analyticity ($N=1$)

$$\mathcal{H} = (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2) \otimes \mathcal{F}$$

For $\psi(\vec{x}) \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$,

$$(U_{el}(\theta)\psi)(\vec{x}) = e^{\frac{3\theta}{2}} \psi(e^{\theta}\vec{x})$$

For $(\varphi_n)_{n=0,1,2,\dots} \in \mathcal{F}$,

$$(U_f(\theta)\varphi_n)(\vec{k}_1, \dots, \vec{k}_n)$$
$$= e^{-3n\theta/2} \varphi_n(e^{-\theta}\vec{k}_1, \dots, e^{-\theta}\vec{k}_n)$$

For $\Lambda < \infty$ (κ_Λ in C^∞),

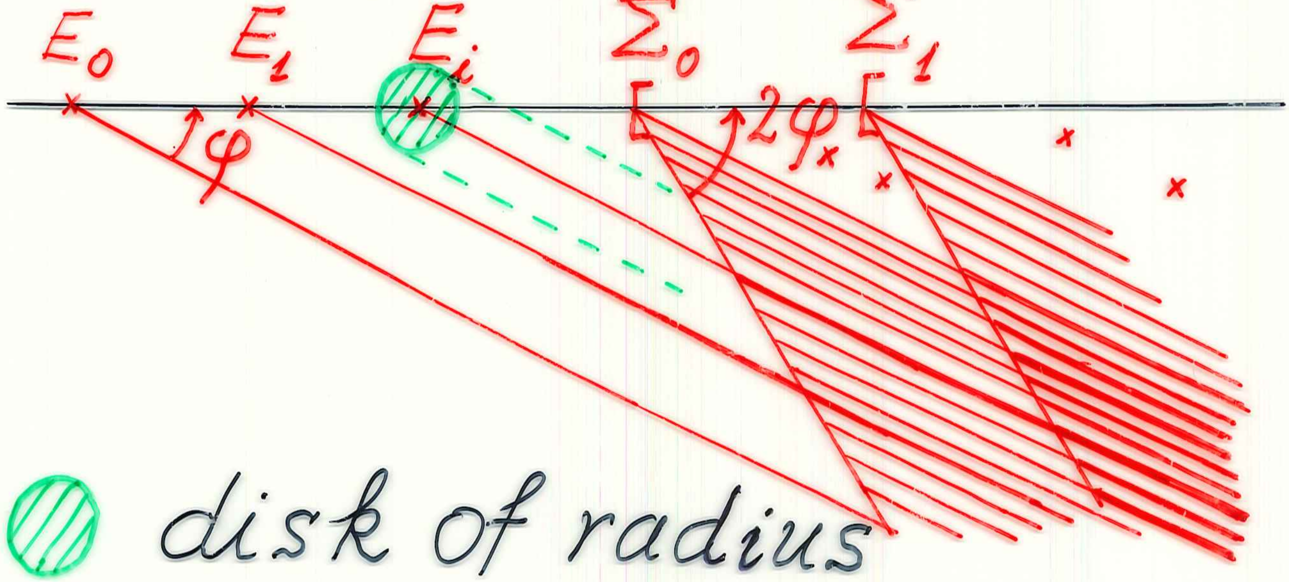
$$H_g(\theta) \equiv U(\theta) H_g U(\theta)^*,$$
$$U(\theta) = U_{el}(\theta) \otimes U_f(\theta),$$

"analytic family" in θ.

$H_f(\theta) = e^{-\theta} H_f$; $-\Delta(\theta) = e^{-2\theta}(-\Delta)$

Set $\theta = i\varphi$, $\varphi > 0$. Then

$spec(H_0(\theta))$:



$\mathcal{D}_{\rho_0} \equiv$ disk of radius

$$\frac{1}{2}\rho_0 \sim g^{2-2\varepsilon}, \quad \varepsilon > 0$$

$$\rho_0 \ll dist(E_i, spec(H_{atom}) \setminus \{E_i\})$$

Interested in

$spec(H_g(\theta)) \cap \mathcal{D}_{\rho_0}$, $\theta = i\varphi$, ($|g|$ small).

Why interesting?

$$\langle \Psi, (z - H_g)^{-1} \Phi \rangle$$

$$= \langle \Psi(\bar{\theta}), (z - H_g(\theta))^{-1} \Phi(\theta) \rangle,$$

for all real θ; $(U(\theta)^* = U(\theta)^{-1}!)$

If Ψ, Φ dilatation-analyt.,
R.S. well def. on complem.
of $\mathrm{spec}(H_g(\theta))$, $\theta = i\varphi \Rightarrow$
analyt. cont. of L.S. in z
from $\{\mathrm{Im}\, z > 0\}$ across cut
to compl. of $\mathrm{spec}(H_g(i\varphi))$.

$\Rightarrow$ Find <u>resonance energies</u>, $E_i(g)$, + nature of cuts.

Tool to analyze spec$(H_g(i\varphi))$:
Iterative perturbation th.
(RG), based on "Feshbach
map" (B-F-S)

$$\mathcal{H},\ P,\ H,\ \overline{P} = \mathbb{1} - P$$

$$H_P = PHP,\ H_{\overline{P}} = \overline{P}H\overline{P} \text{ on } \overline{P}\mathcal{H}.$$

Assume that

$$z \in \mathrm{Res}(H_{\overline{P}}),\ \|R_{\overline{P}}(z)HP\| \ \& \ \|PHR_{\overline{P}}(z)\| < \infty \quad *$$

Feshbach map:

$$(z, H) \mapsto f_{z,P}(H) = \left[H_P - PHR_{\overline{P}}(z) \cdot HP\right]\Big|_{P\mathcal{H}}$$

Theorem. Assume $*$

(i) $z \in \mathrm{spec}(H) \Leftrightarrow z \in \mathrm{spec}(f_{z,P}(H))$

(ii) $z \in \sigma_{pp}(H) \Leftrightarrow z \in \sigma_{pp}(f_{z,P}(H))$

(iii) $[P_1, P_2] = 0 \Rightarrow f_{z,P_1} \circ f_{z,P_2} = f_{z,P_1 \cdot P_2}$

("RG property")

Application to "standard model": $\mathcal{H}$ as above,

$P = P_i^{atom}(i\varphi) \otimes \chi_{H_f \leq \rho_0}$, $\rho_0 \sim g^{2-2\varepsilon}$,

$\varepsilon > 0$, $|g|$ small; $H = H_g(i\varphi)$,

$\varphi > 0$, $\mathcal{D}_{\rho_0}$ as above;

$\mathcal{D}_{\rho_0} \subset \mathrm{Res}(H_{\bar{P}}) \Longrightarrow$

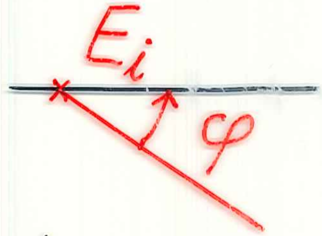
$f_{z,P}(H_g(i\varphi))$ well def.; (i), (ii), (iii)!

$$f_{z,P}(H) = P(E_i + e^{-i\varphi} H_f)P \qquad \mathrm{I}$$

$$+ P W_g P \qquad \mathrm{II}$$

$$+ P W_g \overline{P}(z - H_{\overline{P}})^{-1} \overline{P} W_g P \qquad \mathrm{III}$$

$$\mathrm{I} = E_i + e^{-i\varphi} H_f \text{ on } P\mathcal{H}$$



$$\mathrm{II} \simeq \Delta_1 E_i(g) P, \quad \Delta_1 E_i(g) \in \mathbb{R}$$

$$\mathrm{III} \simeq P W_g \overline{P}(z - H_{0,P})^{-1} \overline{P} W_g P$$

$$\simeq (-i g^2 \Gamma_i + \Delta_2 E_i(g)) P$$

$$\Gamma_i = \sum_{j<i} \int d^3k \, \delta(|\vec{k}| + E_j - E_i) \times$$

$$\times \langle G(\vec{k}) \psi_i, P_j^{atom} G(\vec{k}) \psi_i \rangle$$

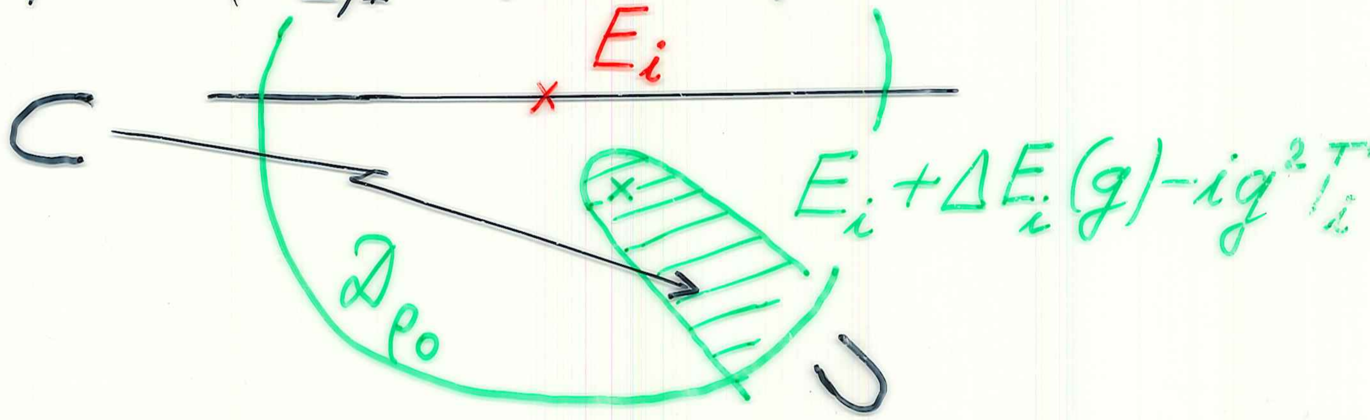
"form factor" of "lin. term"

in W_g; $\quad \Gamma_i > 0$!

"Error terms" are $O(|g|^{3-3\varepsilon})$!

$\Rightarrow$ For $|g|$ small $(0<\varepsilon<\frac{1}{3})$,

$$\mathrm{spec}\left(f_{z,p}(i\varphi)\right) \approx \mathrm{spec}\left(\mathrm{I}+\mathrm{II}+\mathrm{III}\right)$$



By (ii), $\mathrm{spec}\left(H_g(i\varphi)\right) \cap \mathcal{D}_{\rho_0}$!

$$\Rightarrow \left|\left\langle \Psi_i, e^{-itH_g}\Psi_i \right\rangle\right|$$

$$\leq a\, e^{-tg^2\Gamma_i} + O_N(g^4)\, t^{-N}$$

Iterative improvement

(RG); see B-F-S!